The Riesz Representation Theorem in Rudin book can be regarded as a special case of the Caratheodory Extension Theorem

We will show that the Riesz Representation Theorem you have seen in the Rudin book is nothing but a form of the standard Caratheodory Measure Extension approach. Although the approach to the Riesz Representation Theorem in Rudin book might look different from the Caratheodory Extention approach at first glance, they are just equivalent mathematically.

Theorem (Riesz Representation Theorem) Let X be a locally compact Hausdorff space. Let $L\colon C_c(X) \to \mathbb{R}$ be a positive linear functional on $C_c(X)$ which is bounded on all the subspaces $C_c(X)_K$, where K is a compact subset of X and $C_c(X)_K$ is defined as $\{h\colon h \in C(X),\ \mathrm{supp}(h) \subset K\}$. By "$L$ is positive", we mean $L(f) \geq 0$ for any positive function $f \in C_c(X)$. Then there exists a unique measure μ on X, such that μ is complete, both inner regular (when restricted to compact subsets) and outer regular for all measurable sets (see Rudin book for detailed definitions), and for any $g \in C_c(X)$, we have $L(g) = \int_X g \,\mathrm{d}\mu$.

We will just focus on the main thing of the Riesz Representation Theorem, that is, how to derive such a measure. The hints/comments we give here does not closely follow the approach in Rudin book. Instead, it follows the line of the lectures in our class, that is, the process related to Caratheodory Extension Theorem.

Step 0:

The requirement that "L is bounded on all the subspaces $C_c(X)_K$" is redundant. In fact, we will show that any positive linear functional on $C_c(X)$ is automatically bounded on those $C_c(X)_K$, where K is a compact subset.

To achieve this, as X is locally compact and Hausdorff, noting that for any compact set K of X, according to the result in problem 13 of homework 5, which is also a key result used to show the Urysohn's Lemma, we can find an open set U containing K, such that its closure $\overline{U}$ is also compact. In that case, we have (note that $\overline{U}$ is also compact)

$$K \subset U \subset \overline{U} \subset X.$$

With this in mind, by Urysohn's Lemma, we can construct a continuous **positive** function f such that $f\,|_{\overline{U}} = 1$. For any $h \in C_c(X)_K$, we then have $\sup_{x\in K} |h(x)| < +\infty$. It is obvious that $h^+ \leq \sup_{x\in K} |h(x)|$ and $h^- \leq \sup_{x\in K} |h(x)|$. As $K \subset U \subset \overline{U}$, one can check that

$$0 \leq h^+ \leq \sup_{x\in K} |h(x)| \cdot f \quad \text{and} \quad 0 \leq h^- \leq \sup_{x\in K} |h(x)| \cdot f.$$

Note that the functional L is positive. Then we have (why?)

$$0 \leq L(h^+) \leq \sup_{x\in K} |h(x)| \cdot L(f) \quad \text{and} \quad 0 \leq L(h^-) \leq \sup_{x\in K} |h(x)| \cdot L(f).$$

As $L(f) \in \mathbb{R}$, it must be finite. So far, we have proved that

$$|L(h)| = |L(h^+ - h^-)| = |L(h^+) - L(h^-)| \leq |L(h^+)| + |L(h^-)| \leq 2 \cdot \sup_{x\in K} |h(x)| \cdot L(f)$$

for all $h \in C_c(X)_K$.

Step 1: We start with constructing a pre-measure μ_0 on certain "simple" subsets of X, just like defining the pre-measure of $[a, b)$ to be $b-a$ while deriving the Lebesgue measure using the Caratheodory Extension Theorem. In our case, we try to define μ_0 on all the open sets of X. To be precise, for any open set D, we define $\mu_0(D)$ to be

$$\mu_0(D) = \sup\{L(f) \colon f \colon X \to [0,1], \exists \text{ certain compact subset } K \text{ in } D, \text{such that } f\,|_{X-K} = 0\ \}.$$

For this μ_0, easy to check that $\mu_0(\emptyset) = 0$ and $\mu_0(A) \leq \mu_0(B)$ if $A \subset B$.

Step 2: Just like how the outer measure on $\mathbb{R}$ is defined in the constructure of the Lebesgue measure on $\mathbb{R}$, for any subset E of X, we define the outer meaure λ of E to be (different from the one in Rudin book, but can be proved later that these two definitions are the same)

$$\lambda(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(D_i) \colon \text{each } D_i \text{ is open and } E \subset \bigcup_{i=1}^{\infty} D_i \right\}.$$

As is already covered in class, this λ is automatically an outer measure. So far, we are not yet sure that $\lambda = \mu_0$ when restricted to the set of open subsets in X, and that is where the Caratheodory Extension Theorem will come into play.

Step 2.1

If you check the approach in Rudin book, you will realize a slight "difference" in the definition of the outer measure λ. In Rudin book, for any subset E of X, the outer measure, denoted here as λ', is defined as:

$$\lambda'(E) = \inf\ \{\mu_0(D) \colon D \text{ is open and } E \subset D\}.$$

These two definitions of outer measures are equivalent here. That is, for any subset E, we have $\lambda(E) = \lambda'(E)$. To show this, we just need to proof the following claim.

Claim: With the setup as above, for any open subset D and a sequence of open subsets D_i, such that $D \subset \bigcup_{i=1}^{\infty} D_i$, we have

$$\mu_0(D) \leq \sum_{i=1}^{\infty} \mu_0(D_i).$$

Sketchy proof: To prove this claim, we just need to following the definitions. To achieve $\mu_0(D)$, just consider a continuous functions $f \colon X \to [0, 1]$ such that $\mathrm{supp}(f)$ is a subset of certain comapct set K, with $K \subset D$ and with $L(f)$ "close" to $\mu_0(f)$. As f is supported on a compact subset K, $K \subset D \subset \bigcup_{i=1}^{\infty} D_i$, we can find a finite subcovering of K, say $K \subset \bigcup_{i=K}^{\infty} D_i$. As X is locally compact and Hausdorff, we can (check Rudin book for details) write f as the sum of f_i for $1 \leq i \leq K$, where each f_i is continuous and is supported inside D_i. With this in mind, you should be able to finish the rest of the work and show that $\mu_0(D) \leq \sum_{i=1}^{\infty} \mu_0(D_i)$.

Step 2.5

For the μ_0 defined above, show that it is finitely additive. That is, if there are two open sets U and V with $U \cap V = \emptyset$, show that $\mu_0(U \sqcup V) = \mu_0(U) + \mu_0(V)$. If this holds true, then we can get finite additivity on μ_0 simply by induction. It is mostly plain verifications, according to the definition of μ_0. You might want to use the facts like this, "If a compact subset K is in $U \sqcup V$, where both U and V are open sets, then both $K \cap U$ and $K \cap V$ are compact".

Step 2.6

For the μ_0 above, show that it is countably monotonic. If this can be done, combing this with the results we got in Step 2.5, we have proved that μ really extends μ_0. According to the result in Caratheodory Extension Theorem, we just need to show that μ_0 is finitely additive and countably monotone.

As for finite additiveness, for any finite disjoint open sets $D_1, \cdots, D_n$, we need to show that

$$\mu_0\left(\bigsqcup_{i=1}^{n} D_i\right) = \sum_{i=1}^{n} \mu_0(D_i).$$

It is trivial to show that $\sum_{i=1}^{n} \mu_0(D_i) \leq \mu_0\left(\bigsqcup_{i=1}^{n} D_i\right)$ using definitions (why?). It just remains to show

$$\mu_0\left(\bigsqcup_{i=1}^{n} D_i\right) \leq \sum_{i=1}^{n} \mu_0(D_i),$$

which is mainly about checking against definitions. That is, according to the definition of μ_0, $\mu_0\left(\bigsqcup_{i=1}^{n} D_i\right)$ equals .... . Some math you might want to use is 1) every open covering of a compact set contains a finite subcovering, and 2) on a locally compact Hausdorff space X (which implies that X is paracompact), a continuous function f which is defined/supported on a finite union of open sets, say, $\bigcup_{i=1}^{n} U_i$, can be written as the sum of functions f_i, such that each f_i is supported on the corresponding U_i only.

Step 3:

As like the standard process we had done in class, this above defined λ might not be a measure on $\mathcal{P}(X)$, but it will be a measure when restricted to a subset $\mathcal{M}$ of $\mathcal{P}(X)$. Here we have $E \subset \mathcal{M}$ if

$$\lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c) \quad \text{for all } A \subset X.$$

As is covered in the Caratheodory Extension Theorem, when restriected on $\mathcal{M}$, the λ is really a measure. We use μ to denote this measure. That is,

$$\mu = \lambda\,|_{\mathcal{M}}\,.$$

Now, it remains to show some important properties of this measure $(\mu, \mathcal{M})$.

Step 4:

Check that each Borel set is measurable. That is, for any Borel set D, we have

$$\lambda(A) = \lambda(A \cap D) + \lambda(A \cap D^c)$$

for all subset A of $\mathbb{R}$.

In fact, recalling that those measurable sets (in the sense above) form a σ-algebra (check your notes on Caratheodory Extension Theorem related stuff for details), we just need to show that every open set is measurable. That is, for any open set E and for any subset A of $\mathbb{R}$, we have

$$\lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c).$$

We can prove the following claim first.

Claim: For any set A and any $\epsilon > 0$, there exists an open set U such that $U \supset A$, and $|\lambda(A \cap B) - \lambda(U \cap B)| < \epsilon$ for all the subsets B.

Sketch of the proof: We can use the definition of λ to find an open set U such that $A \subset U$ and $|\lambda(A) - \lambda(U)| < \epsilon$. Note that the outer measure λ is subadditive, and we can check that this U is the desired one.

With that claim in mind, we just need to show that for any given open set E and any open set U of $\mathbb{R}$, we have

$$\lambda(U) = \lambda(U \cap E) + \lambda(U \cap E^c).$$

If that is true, with the claim above, we can show that $\lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c)$ for any subset A. Thus any open set U is measurable.

In fact, as the outer measure μ is subadditive, we just need to check that

$$\lambda(U \cap E) + \lambda(U \cap E^c) \leq \lambda(U).$$

This is relatively easy to check. Just note that $U \cap E$ is also open, and we can find, for any given $\epsilon > 0$, a compact subset K in $U \cap E$, such that there exists a continuous function $f\colon X \to [0,1]$ such that the support of f is inside K, and

$$\lambda(U \cap E) - \epsilon \leq L(f) \leq \mu_0(U \cap E) = \lambda(U \cap E).$$

As X is Hausdorff, the compact subset K is also closed. Thus K^c is open. As $U \cap K^c$ is open and $U \cap K^c \supset U \cap E^c$, according to the definition of $\lambda(U \cap E^c)$, we can find a compact subset F in $U \cap K^c$ and a continuous function $g\colon X \to [0,1]$ such that the support of g is inside F and

$$\lambda(U \cap E^c) - \epsilon \leq L(g) \leq \mu_0(U \cap E^c) = \lambda(U \cap E^c).$$

Now, it should be easy to show that $\lambda(U \cap E) + \lambda(U \cap E^c) \leq \lambda(U)$.

Step 5:

For each compact subset K, show that $\mu(K) < +\infty$. We can borrow the proof of the same fact from Rudin book, and it will work without problem. That is because the proof only needs the fact that as L is a positive linear functional, it is automatically bounded on the set of continuous functions with any **given** compact support. A sketchy proof of this fact can be found in Step 0.

Also, following the sketchy proof in Step 0, we can directly deduce the that $\mu(K) < +\infty$ for any compact subset D, without having to borrowing anything from the proof on Riesz Representation Theorem in Rudin book.

Note that we can use $\mu(K)$ as we can now safely claim that every compact set K is measurable in the above sense as in Step 3. This is because that in this Hausdorff space X, every compact space is closed, thus lies in the σ-algebra generated by the open sets.

Step 6:

In the proof of the Riesz Representation Theorem in Rudin book, the set of measurable subsets of X is definely "differently" compared with the definition here of $\mathcal{M}$ in Step 3. This difference is not essential. These two definitions of "measurability" are equivalent, as we will show in this step.

For simplicity, we just assume the total space X is compact. If not, with slightly more work, parallel arguments will get the job done.

First, if a subset D is measurable in the sense as the part of Riesz Representation Theorem of Rudin book, we have

$$\underline{\lambda}(D) = \overline{\lambda}(D),$$

where

$$\underline{\lambda}(D) = \sup\{\lambda(K)\colon K \text{ is compact and } K \subset D\}$$

and

$$\overline{\lambda}(D) = \inf\{\lambda(U)\colon U \text{ is open and } U \supset D\}.$$

Note that this $\overline{\lambda}$ is the same as the λ we defined above. We use this notation $\overline{\lambda}$ to better indicate its relation with $\underline{\lambda}$.

Now, assume a subset D is measurable in the sense of Rudin book, we will show that it is also measurable in the sense of our defintion above. That is, for any subset A, we have

$$\overline{\lambda}(A) = \overline{\lambda}(A \cap D) + \overline{\lambda}(A \cap D^c).$$

As $\overline{\lambda}$ is an outer measure, we just need to show

$$\overline{\lambda}(A \cap D) + \overline{\lambda}(A \cap D^c) \leq \overline{\lambda}(A).$$

Note that, in general, we shall not expect any of the following three equations to hold true: $\overline{\lambda}(A \cap D) = \underline{\lambda}(A \cap D)$, $\overline{\lambda}(A \cap D^c) = \underline{\lambda}(A \cap D^c)$ and $\overline{\lambda}(A) = \underline{\lambda}(A)$.

As D is measurable in the sense of Rudin book, and as X is compact, it follows (why?) immediately that D^c is also measurable in the sense of Rudin book. Also, it is proved in Rudin book that every open set is measurable in the sense of Rudin book. That is, for every open set E in X, we have $\overline{\lambda}(E) = \underline{\lambda}(E)$. Besides, if two subsets are measurable in the sense of Rudin book, it is proved in Rudin book that their intersection is also measurable in the sense of the Rudin book.

Now, back to what we need to do: prove that $\overline{\lambda}(A \cap D) + \overline{\lambda}(A \cap D^c) \leq \overline{\lambda}(A)$.

Key step: Following the observation in Step 4, we just need to prove $\overline{\lambda}(A \cap D) + \overline{\lambda}(A \cap D^c) \leq \overline{\lambda}(A)$ in case A is an open subset.

Note that A, D and D^c are all measurable in the sense of Rudin book, thus so is A, $A \cap D$ and $A \cap D^c$. Then we have

$$\overline{\lambda}(A \cap D) + \overline{\lambda}(A \cap D^c) \leq \overline{\lambda}(A) \quad \Longleftrightarrow \quad \underline{\lambda}(A \cap D) + \underline{\lambda}(A \cap D^c) \leq \underline{\lambda}(A)$$

Note that $(A \cap D) \bigcap (A \cap D^c) = \emptyset$ and $(A \cap D) \bigcup (A \cap D^c) = A$, from the definition of $\underline{\lambda}$, it follows (why?) that

$$\underline{\lambda}(A \cap D) + \underline{\lambda}(A \cap D^c) \leq \underline{\lambda}(A),$$

for every open set A (thus eventually for every subset A. See Step 4 for details).

So far, we have shown that if a subset is measurable in the sense of the Rudin book, then it is measurable in the sense of our Caratheodory Extension Theorem approach here, as stated in Step 3.

Now, we will show that if a subset D is measurable in the sense of our Caratheodory Extension Theorem approach as in Step 3, then it is measurable in the sense of the Rudin book (i.e, $\overline{\lambda}(D) = \underline{\lambda}(D)$).

From the definition of $\overline{\lambda}$ and $\underline{\lambda}$, we have the following claim, whose proof is just checking against definitions.

<u>Claim:</u> For *any* subset H of X, we have $\overline{\lambda}(H) + \underline{\lambda}(H^c) = \lambda(X)$.

Now, the proof. Assume that E is measurable in the sense of Step 3. Will show that $\overline{\lambda}(D) = \underline{\lambda}(D)$.

As E is measurable in the sense of Step 3, we have

$$\overline{\lambda}(X) \geq \overline{\lambda}(D) + \overline{\lambda}(D^c).$$

As X is assumed to be compact, we can prove (why?) that

$$\underline{\lambda}(X) = \overline{\lambda}(X) = \lambda(X).$$

To show that $\underline{\lambda}(D) = \overline{\lambda}(D)$, we just need to show $\underline{\lambda}(X) \leq \underline{\lambda}(D) + \underline{\lambda}(D^c)$. In fact, if so, then

$$\begin{aligned} \overline{\lambda}(X) &\geq \overline{\lambda}(D) + \overline{\lambda}(D^c) \\ &\geq \underline{\lambda}(D) + \underline{\lambda}(D^c) \\ &\geq \underline{\lambda}(X) \\ &= \overline{\lambda}(X). \end{aligned}$$

Thus it follows that $\overline{\lambda}(D) = \underline{\lambda}(D)$, which finishes the proof. In fact, the reasoning above also indicates that $\overline{\lambda}(D^c) = \underline{\lambda}(D^c)$.

It only remains to show that $\underline{\lambda}(X) \leq \underline{\lambda}(D) + \underline{\lambda}(D^c)$. A stupid proof is like this: According to the assumption, we have $\overline{\lambda}(X) \geq \overline{\lambda}(D) + \overline{\lambda}(D^c)$. According to the definition of $\overline{\lambda}$, for any $\epsilon > 0$, we can find open sets E_1 and E_2, such that $E_1 \supset D$, $E_2 \supset D^c$, and $\lambda(E_1) + \lambda(E_2) \leq \lambda(X) + \epsilon$. As X is compact, we know that E_1^c and E_2^c are compact subsets in X. Besides, $E_1^c \subset D^c$ and $E_2^c \subset D$. According to the Claim above, we have

$$\begin{aligned}
\underline{\lambda}(E_1^c) + \underline{\lambda}(E_2^c) &= \lambda(X) - \overline{\lambda}(E_1) + \lambda(X) - \overline{\lambda}(E_2) \\
&= \lambda(X) + \big(\lambda(X) - \overline{\lambda}(E_1) - \overline{\lambda}(E_2)\big) \\
&\geq \lambda(X) - \epsilon \\
&= \underline{\lambda}(X) - \epsilon.
\end{aligned}$$

Let $\epsilon \to 0$, and we are done.

Note: A smarter proof can be done as follows: As $\overline{\lambda}(X) \geq \overline{\lambda}(D) + \overline{\lambda}(D^c)$, according to the Claim above, we have

$$\lambda(X) - \overline{\lambda}(\emptyset) \leq (\lambda(X) - \overline{\lambda}(D^c)) + (\lambda(X) - \overline{\lambda}(D)).$$

Thus

$$\overline{\lambda}(D) + \overline{\lambda}(D^c) \leq \lambda(X) = \overline{\lambda}(X).$$